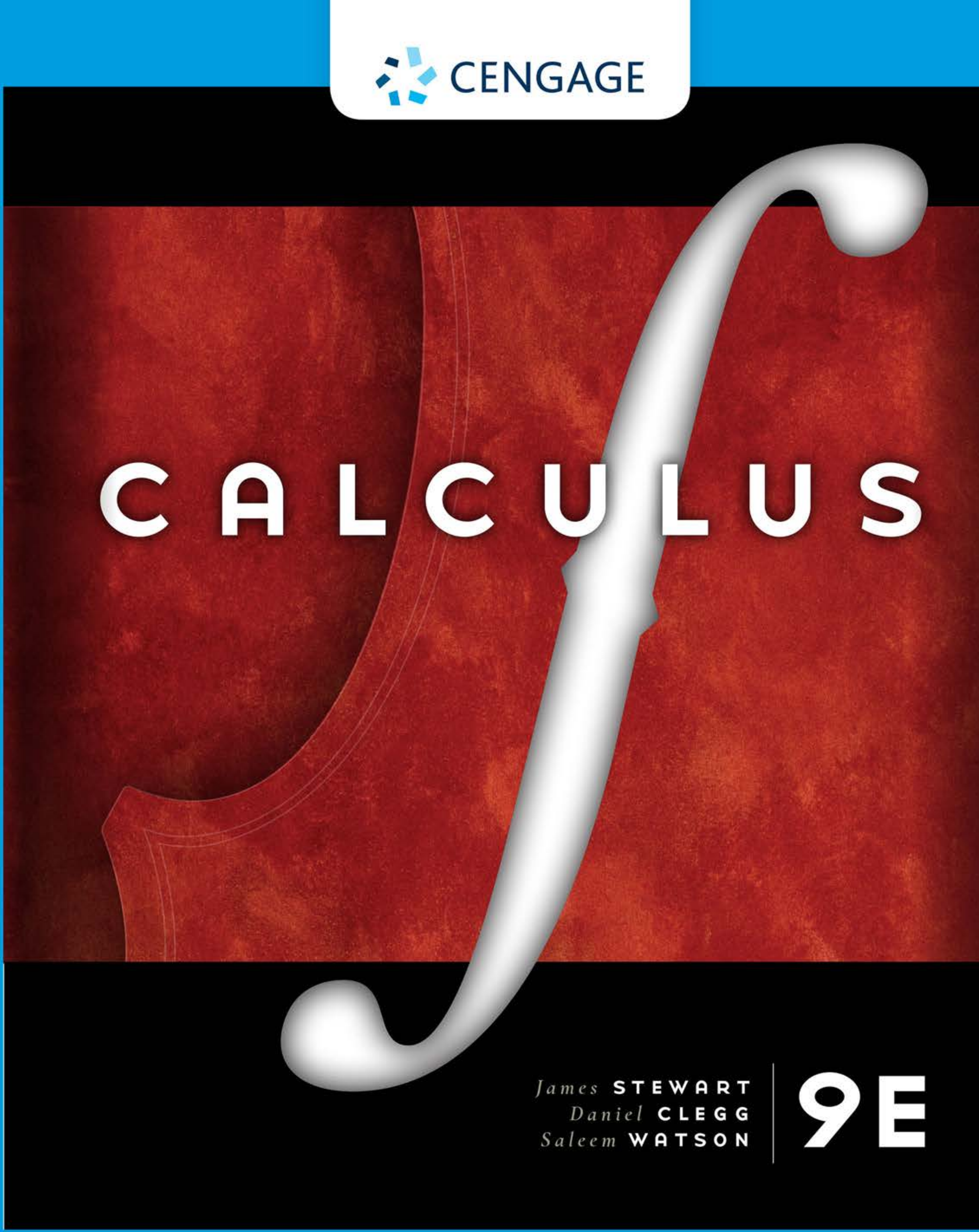
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James STEWART
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REFERENCE page 1

ALGEBRA

Arithmetic Operations

$$a(b + c) = ab + ac \qquad \frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

$$\frac{a + c}{b} = \frac{a}{b} + \frac{c}{b} \qquad \frac{\dfrac{a}{b}}{\dfrac{c}{d}} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$$

Exponents and Radicals

$$x^m x^n = x^{m+n} \qquad \frac{x^m}{x^n} = x^{m-n}$$

$$(x^m)^n = x^{mn} \qquad x^{-n} = \frac{1}{x^n}$$

$$(xy)^n = x^n y^n \qquad \left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$$

$$x^{1/n} = \sqrt[n]{x} \qquad x^{m/n} = \sqrt[n]{x^m} = \left(\sqrt[n]{x}\right)^m$$

$$\sqrt[n]{xy} = \sqrt[n]{x}\,\sqrt[n]{y} \qquad \sqrt[n]{\frac{x}{y}} = \frac{\sqrt[n]{x}}{\sqrt[n]{y}}$$

Factoring Special Polynomials

$$x^2 - y^2 = (x + y)(x - y)$$

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$$

$$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$$

Binomial Theorem

$$(x + y)^2 = x^2 + 2xy + y^2 \qquad (x - y)^2 = x^2 - 2xy + y^2$$

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x - y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$$

$$(x + y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2}x^{n-2}y^2 + \cdots + \binom{n}{k}x^{n-k}y^k + \cdots + nxy^{n-1} + y^n$$

where $\displaystyle\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{1 \cdot 2 \cdot 3 \cdot \cdots \cdot k}$

Quadratic Formula

If $ax^2 + bx + c = 0$, then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Inequalities and Absolute Value

If $a < b$ and $b < c$, then $a < c$.

If $a < b$, then $a + c < b + c$.

If $a < b$ and $c > 0$, then $ca < cb$.

If $a < b$ and $c < 0$, then $ca > cb$.

If $a > 0$, then

$|x| = a$ means $x = a$ or $x = -a$

$|x| < a$ means $-a < x < a$

$|x| > a$ means $x > a$ or $x < -a$

GEOMETRY

Geometric Formulas

Formulas for area A, circumference C, and volume V:

Triangle

$A = \frac{1}{2}bh$

$= \frac{1}{2}ab\sin\theta$

Circle

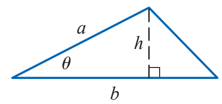
$A = \pi r^2$

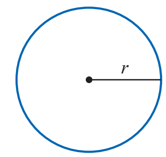
$C = 2\pi r$

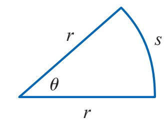
Sector of Circle

$A = \frac{1}{2}r^2\theta$

$s = r\theta$ (θ in radians)







Sphere

$V = \frac{4}{3}\pi r^3$

$A = 4\pi r^2$

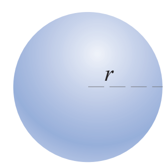
Cylinder

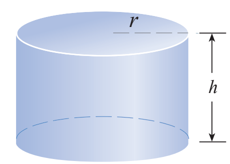
$V = \pi r^2 h$

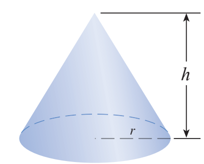
Cone

$V = \frac{1}{3}\pi r^2 h$

$A = \pi r\sqrt{r^2 + h^2}$







Distance and Midpoint Formulas

Distance between $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Midpoint of $\overline{P_1P_2}$: $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$

Lines

Slope of line through $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Point-slope equation of line through $P_1(x_1, y_1)$ with slope m:

$$y - y_1 = m(x - x_1)$$

Slope-intercept equation of line with slope m and y-intercept b:

$$y = mx + b$$

Circles

Equation of the circle with center (h, k) and radius r:

$$(x - h)^2 + (y - k)^2 = r^2$$

TRIGONOMETRY

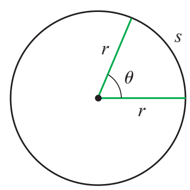
Angle Measurement

π radians $= 180°$

$1° = \dfrac{\pi}{180}$ rad $\qquad$ 1 rad $= \dfrac{180°}{\pi}$

$s = r\theta$

(θ in radians)

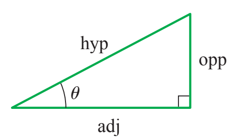


Right Angle Trigonometry

$\sin\theta = \dfrac{\text{opp}}{\text{hyp}}$ $\qquad$ $\csc\theta = \dfrac{\text{hyp}}{\text{opp}}$

$\cos\theta = \dfrac{\text{adj}}{\text{hyp}}$ $\qquad$ $\sec\theta = \dfrac{\text{hyp}}{\text{adj}}$

$\tan\theta = \dfrac{\text{opp}}{\text{adj}}$ $\qquad$ $\cot\theta = \dfrac{\text{adj}}{\text{opp}}$

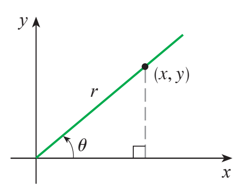


Trigonometric Functions

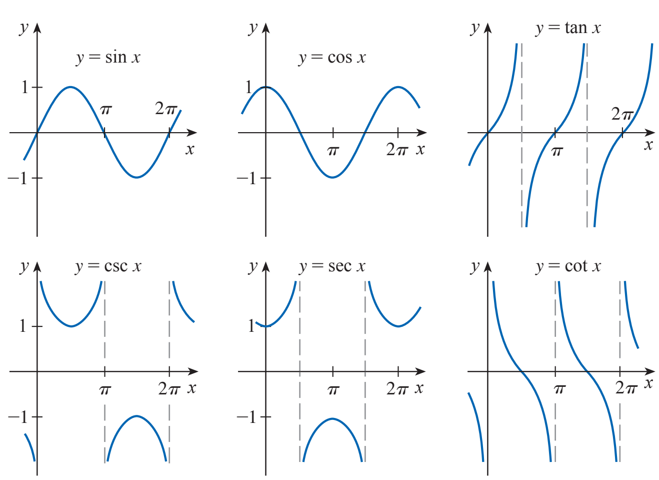
$\sin\theta = \dfrac{y}{r}$ $\qquad$ $\csc\theta = \dfrac{r}{y}$

$\cos\theta = \dfrac{x}{r}$ $\qquad$ $\sec\theta = \dfrac{r}{x}$

$\tan\theta = \dfrac{y}{x}$ $\qquad$ $\cot\theta = \dfrac{x}{y}$



Graphs of Trigonometric Functions



Trigonometric Functions of Important Angles

θ	radians	$\sin\theta$	$\cos\theta$	$\tan\theta$
0°	0	0	1	0
30°	$\pi/6$	$1/2$	$\sqrt{3}/2$	$\sqrt{3}/3$
45°	$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$	1
60°	$\pi/3$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$
90°	$\pi/2$	1	0	—

Fundamental Identities

$\csc\theta = \dfrac{1}{\sin\theta}$ $\qquad$ $\sec\theta = \dfrac{1}{\cos\theta}$

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ $\qquad$ $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

$\cot\theta = \dfrac{1}{\tan\theta}$ $\qquad$ $\sin^2\theta + \cos^2\theta = 1$

$1 + \tan^2\theta = \sec^2\theta$ $\qquad$ $1 + \cot^2\theta = \csc^2\theta$

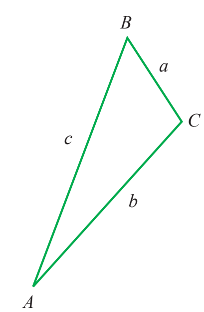
$\sin(-\theta) = -\sin\theta$ $\qquad$ $\cos(-\theta) = \cos\theta$

$\tan(-\theta) = -\tan\theta$ $\qquad$ $\sin\left(\dfrac{\pi}{2} - \theta\right) = \cos\theta$

$\cos\left(\dfrac{\pi}{2} - \theta\right) = \sin\theta$ $\qquad$ $\tan\left(\dfrac{\pi}{2} - \theta\right) = \cot\theta$

The Law of Sines

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$



The Law of Cosines

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

Addition and Subtraction Formulas

$$\sin(x + y) = \sin x\cos y + \cos x\sin y$$

$$\sin(x - y) = \sin x\cos y - \cos x\sin y$$

$$\cos(x + y) = \cos x\cos y - \sin x\sin y$$

$$\cos(x - y) = \cos x\cos y + \sin x\sin y$$

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x\tan y}$$

$$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x\tan y}$$

Double-Angle Formulas

$$\sin 2x = 2\sin x\cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$

Half-Angle Formulas

$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$

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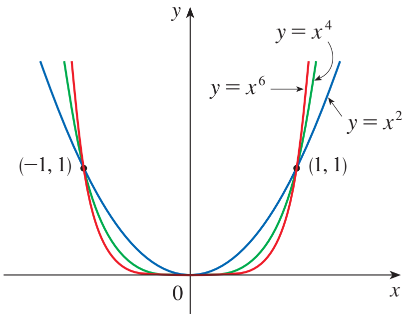
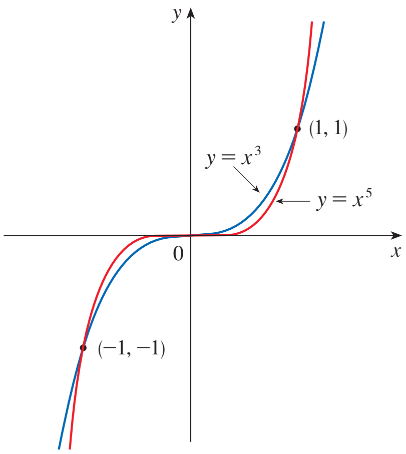
SPECIAL FUNCTIONS

Power Functions $f(x) = x^a$

(i) $f(x) = x^n$, n a positive integer

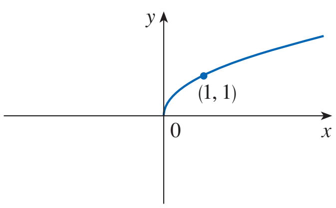


n even

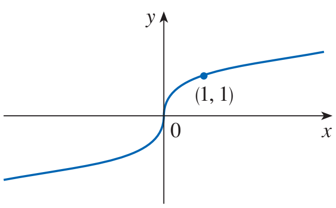


n odd

(ii) $f(x) = x^{1/n} = \sqrt[n]{x}$, n a positive integer

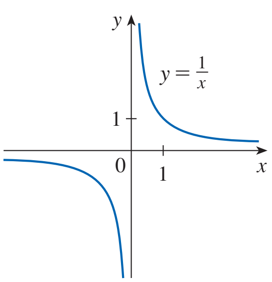


$f(x) = \sqrt{x}$



$f(x) = \sqrt[3]{x}$

(iii) $f(x) = x^{-1} = \dfrac{1}{x}$

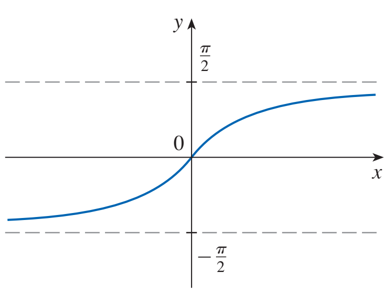


Inverse Trigonometric Functions

$\arcsin x = \sin^{-1}x = y \iff \sin y = x$ and $-\dfrac{\pi}{2} \leqslant y \leqslant \dfrac{\pi}{2}$

$\arccos x = \cos^{-1}x = y \iff \cos y = x$ and $0 \leqslant y \leqslant \pi$

$\arctan x = \tan^{-1}x = y \iff \tan y = x$ and $-\dfrac{\pi}{2} < y < \dfrac{\pi}{2}$



$y = \tan^{-1}x = \arctan x$

$$\lim_{x \to -\infty} \tan^{-1}x = -\frac{\pi}{2}$$

$$\lim_{x \to \infty} \tan^{-1}x = \frac{\pi}{2}$$

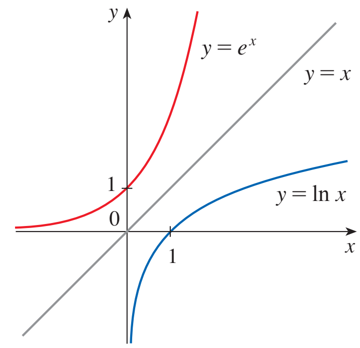
SPECIAL FUNCTIONS

Exponential and Logarithmic Functions

$\log_b x = y \iff b^y = x$

$\ln x = \log_e x, \quad \text{where} \quad \ln e = 1$

$\ln x = y \iff e^y = x$



Cancellation Equations

$\log_b(b^x) = x \qquad b^{\log_b x} = x$

$\ln(e^x) = x \qquad e^{\ln x} = x$

Laws of Logarithms

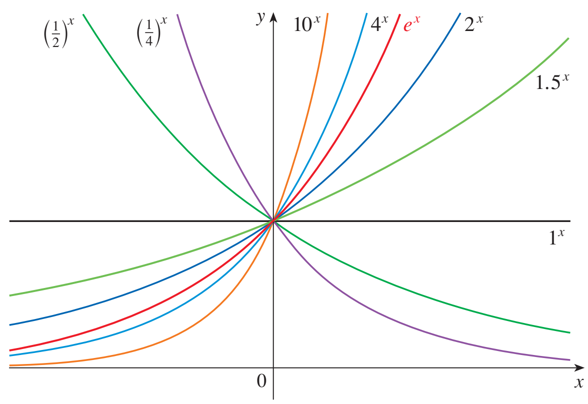
1. $\log_b(xy) = \log_b x + \log_b y$

2. $\log_b\left(\dfrac{x}{y}\right) = \log_b x - \log_b y$

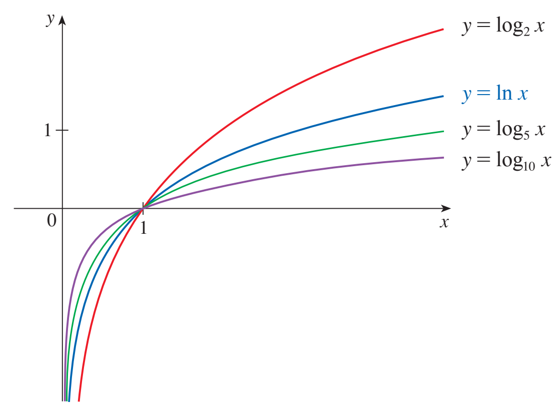
3. $\log_b(x^r) = r\log_b x$

$\lim_{x\to-\infty} e^x = 0 \qquad \lim_{x\to\infty} e^x = \infty$

$\lim_{x\to 0^+} \ln x = -\infty \qquad \lim_{x\to\infty} \ln x = \infty$



Exponential functions



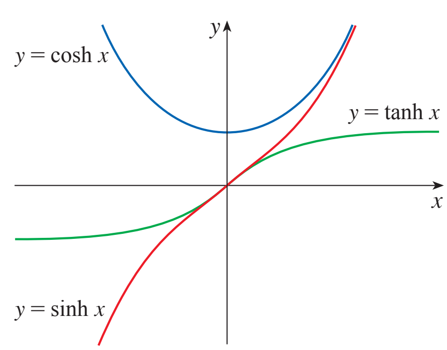
Logarithmic functions

Hyperbolic Functions

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch} x = \frac{1}{\sinh x}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x}$$

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{\cosh x}{\sinh x}$$



Inverse Hyperbolic Functions

$y = \sinh^{-1}x \iff \sinh y = x$

$y = \cosh^{-1}x \iff \cosh y = x \quad \text{and} \quad y \geqslant 0$

$y = \tanh^{-1}x \iff \tanh y = x$

$$\sinh^{-1}x = \ln\left(x + \sqrt{x^2 + 1}\right)$$

$$\cosh^{-1}x = \ln\left(x + \sqrt{x^2 - 1}\right)$$

$$\tanh^{-1}x = \tfrac{1}{2}\ln\left(\frac{1 + x}{1 - x}\right)$$

CALCULUS

NINTH EDITION

JAMES STEWART

McMASTER UNIVERSITY
AND
UNIVERSITY OF TORONTO

DANIEL CLEGG

PALOMAR COLLEGE

SALEEM WATSON

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James Stewart, Daniel Clegg, Saleem Watson

Product Director: Mark Santee

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Art Directors: Angela Sheehan, Vernon Boes

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Contents

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Appendixes A1

Preface

A great discovery solves a great problem but there is a grain of discovery in the solution of any problem. Your problem may be modest; but if it challenges your curiosity and brings into play your inventive faculties, and if you solve it by your own means, you may experience the tension and enjoy the triumph of discovery.

GEORGE POLYA

The art of teaching, Mark Van Doren said, is the art of assisting discovery. In this Ninth Edition, as in all of the preceding editions, we continue the tradition of writing a book that, we hope, assists students in discovering calculus—both for its practical power and its surprising beauty. We aim to convey to the student a sense of the utility of calculus as well as to promote development of technical ability. At the same time, we strive to give some appreciation for the intrinsic beauty of the subject. Newton undoubtedly experienced a sense of triumph when he made his great discoveries. We want students to share some of that excitement.

The emphasis is on understanding concepts. Nearly all calculus instructors agree that conceptual understanding should be the ultimate goal of calculus instruction; to implement this goal we present fundamental topics graphically, numerically, algebraically, and verbally, with an emphasis on the relationships between these different representations. Visualization, numerical and graphical experimentation, and verbal descriptions can greatly facilitate conceptual understanding. Moreover, conceptual understanding and technical skill can go hand in hand, each reinforcing the other.

We are keenly aware that good teaching comes in different forms and that there are different approaches to teaching and learning calculus, so the exposition and exercises are designed to accommodate different teaching and learning styles. The features (including projects, extended exercises, principles of problem solving, and historical insights) provide a variety of enhancements to a central core of fundamental concepts and skills. Our aim is to provide instructors and their students with the tools they need to chart their own paths to discovering calculus.

Alternate Versions

The Stewart *Calculus* series includes several other calculus textbooks that might be preferable for some instructors. Most of them also come in single variable and multivariable versions.

- *Calculus: Early Transcendentals*, Ninth Edition, is similar to the present textbook except that the exponential, logarithmic, and inverse trigonometric functions are covered in the first semester.

- *Essential Calculus*, Second Edition, is a much briefer book (840 pages), though it contains almost all of the topics in *Calculus*, Ninth Edition. The relative brevity is achieved through briefer exposition of some topics and putting some features on the website.

- *Essential Calculus: Early Transcendentals*, Second Edition, resembles *Essential Calculus*, but the exponential, logarithmic, and inverse trigonometric functions are covered in Chapter 3.
- *Calculus: Concepts and Contexts*, Fourth Edition, emphasizes conceptual understanding even more strongly than this book. The coverage of topics is not encyclopedic and the material on transcendental functions and on parametric equations is woven throughout the book instead of being treated in separate chapters.
- *Brief Applied Calculus* is intended for students in business, the social sciences, and the life sciences.
- *Biocalculus: Calculus, Probability and Statistics for the Life Sciences* is intended to show students in the life sciences how calculus relates to biology. It includes three chapters covering probability and statistics.

What's New in the Ninth Edition?

The overall structure of the text remains largely the same, but we have made many improvements that are intended to make the Ninth Edition even more usable as a teaching tool for instructors and as a learning tool for students. The changes are a result of conversations with our colleagues and students, suggestions from users and reviewers, insights gained from our own experiences teaching from the book, and from the copious notes that James Stewart entrusted to us about changes that he wanted us to consider for the new edition. In all the changes, both small and large, we have retained the features and tone that have contributed to the success of this book.

- More than 20% of the exercises are new:

  Basic exercises have been added, where appropriate, near the beginning of exercise sets. These exercises are intended to build student confidence and reinforce understanding of the fundamental concepts of a section. (See, for instance, Exercises 7.3.1–4, 9.1.1–5, 11.4.3–6.)

  Some new exercises include graphs intended to encourage students to understand how a graph facilitates the solution of a problem; these exercises complement subsequent exercises in which students need to supply their own graph. (See Exercises 5.2.1–4, Exercises 10.4.43–46 as well as 53–54, 15.5.1–2, 15.6.9–12, 16.7.15 and 24, 16.8.9 and 13.)

  Some exercises have been structured in two stages, where part (a) asks for the setup and part (b) is the evaluation. This allows students to check their answer to part (a) before completing the problem. (See Exercises 5.1.1–4, 5.3.3–4, 15.2.7–10.)

  Some challenging and extended exercises have been added toward the end of selected exercise sets (such as Exercises 5.2.87, 9.3.56, 11.2.79–81, and 11.9.47).

  Titles have been added to selected exercises when the exercise extends a concept discussed in the section. (See, for example, Exercises 3.4.64, 10.1.55–57, and 15.2.80–81.)

  Some of our favorite new exercises are 1.3.71, 2.6.63, 3.5.41–44, 5.2.79, 5.5.18, 6.4.99 (also 6.4*.67), 10.5.69, 15.1.38, and 15.4.3–4. In addition, Problem 14 in the Problems Plus following Chapter 5 and Problem 4 in the Problems Plus following Chapter 15 are interesting and challenging.

- New examples have been added, and additional steps have been added to the solutions of some existing examples. (See, for instance, Example 2.1.5, Example 5.3.5, Example 10.1.5, Examples 14.8.1 and 14.8.4, and Example 16.3.4.)
- Several sections have been restructured and new subheads added to focus the organization around key concepts. (Good illustrations of this are Sections 1.6, 11.1, 11.2, and 14.2.)
- Many new graphs and illustrations have been added, and existing ones updated, to provide additional graphical insights into key concepts.
- A few new topics have been added and others expanded (within a section or in extended exercises) that were requested by reviewers. (Examples include a subsection on torsion in Section 13.3, symmetric difference quotients in Exercise 2.1.60, and improper integrals of more than one type in Exercises 7.8.65–68.)
- New projects have been added and some existing projects have been updated. (For instance, see the Discovery Project following Section 12.2, *The Shape of a Hanging Chain*.)
- Alternating series and absolute convergence are now covered in one section (11.5).
- The chapter on Second-Order Differential Equations, as well as the associated appendix section on complex numbers, has been moved to the website.

Features

Each feature is designed to complement different teaching and learning practices. Throughout the text there are historical insights, extended exercises, projects, problem-solving principles, and many opportunities to experiment with concepts by using technology. We are mindful that there is rarely enough time in a semester to utilize all of these features, but their availability in the book gives the instructor the option to assign some and perhaps simply draw attention to others in order to emphasize the rich ideas of calculus and its crucial importance in the real world.

Conceptual Exercises

The most important way to foster conceptual understanding is through the problems that the instructor assigns. To that end we have included various types of problems. Some exercise sets begin with requests to explain the meanings of the basic concepts of the section (see, for instance, the first few exercises in Sections 1.5, 1.8, 11.2, 14.2, and 14.3) and most exercise sets contain exercises designed to reinforce basic understanding (such as Exercises 1.8.3–10, 4.5.1–8, 5.1.1–4, 7.3.1–4, 9.1.1–5, and 11.4.3–6). Other exercises test conceptual understanding through graphs or tables (see Exercises 2.1.17, 2.2.34–36, 2.2.45–50, 9.1.23–25, 10.1.30–33, 13.2.1–2, 13.3.37–43, 14.1.41–44, 14.3.2, 14.3.4–6, 14.6.1–2, 14.7.3–4, 15.1.6–8, 16.1.13–22, 16.2.19–20, and 16.3.1–2).

Many exercises provide a graph to aid in visualization (see for instance Exercises 5.2.1–4, 10.4.43–46, 15.5.1–2, 15.6.9–12, and 16.7.24). Another type of exercise uses verbal descriptions to gauge conceptual understanding (see Exercises 1.8.12, 2.2.64, 3.3.65–66, and 7.8.79). In addition, all the review sections begin with a Concept Check and a True-False Quiz.

We particularly value problems that combine and compare graphical, numerical, and algebraic approaches (see Exercises 2.7.27, 3.4.33–34, and 9.4.4).

Graded Exercise Sets

Each exercise set is carefully graded, progressing from basic conceptual exercises, to skill-development and graphical exercises, and then to more challenging exercises that often extend the concepts of the section, draw on concepts from previous sections, or involve applications or proofs.

Real-World Data

Real-world data provide a tangible way to introduce, motivate, or illustrate the concepts of calculus. As a result, many of the examples and exercises deal with functions defined by such numerical data or graphs. These real-world data have been obtained by contacting companies and government agencies as well as researching on the Internet and in libraries. See, for instance, Figure 1 in Section 1.1 (seismograms from the Northridge earthquake), Exercise 2.2.34 (number of cosmetic surgeries), Exercise 4.1.12 (velocity of the space shuttle *Endeavour*), Exercise 4.4.73 (power consumption in the New England states), Example 3 in Section 14.4 (the heat index), Figure 1 in Section 14.6 (temperature contour map), Example 9 in Section 15.1 (snowfall in Colorado), and Figure 1 in Section 16.1 (velocity vector fields of wind in San Francisco Bay).

Projects

One way of involving students and making them active learners is to have them work (perhaps in groups) on extended projects that give a feeling of substantial accomplishment when completed. There are three kinds of projects in the text.

Applied Projects involve applications that are designed to appeal to the imagination of students. The project after Section 9.5 asks whether a ball thrown upward takes longer to reach its maximum height or to fall back to its original height (the answer might surprise you). The project after Section 14.8 uses Lagrange multipliers to determine the masses of the three stages of a rocket so as to minimize the total mass while enabling the rocket to reach a desired velocity.

Discovery Projects anticipate results to be discussed later or encourage discovery through pattern recognition (see the project following Section 7.6, which explores patterns in integrals). Other discovery projects explore aspects of geometry: tetrahedra (after Section 12.4), hyperspheres (after Section 15.6), and intersections of three cylinders (after Section 15.7). Additionally, the project following Section 12.2 uses the geometric definition of the derivative to find a formula for the shape of a hanging chain. Some projects make substantial use of technology; the one following Section 10.2 shows how to use Bézier curves to design shapes that represent letters for a laser printer.

Writing Projects ask students to compare present-day methods with those of the founders of calculus—Fermat's method for finding tangents, for instance, following Section 2.1. Suggested references are supplied.

More projects can be found in the *Instructor's Guide*. There are also extended exercises that can serve as smaller projects. (See Exercise 3.7.53 on the geometry of beehive cells, Exercise 5.2.87 on scaling solids of revolution, or Exercise 9.3.56 on the formation of sea ice.)

Problem Solving

Students usually have difficulties with problems that have no single well-defined procedure for obtaining the answer. As a student of George Polya, James Stewart experienced first-hand Polya's delightful and penetrating insights into the process of problem solving. Accordingly, a modified version of Polya's four-stage problem-solving strategy is presented following Chapter 1 in Principles of Problem Solving.

These principles are applied, both explicitly and implicitly, throughout the book. Each of the other chapters is followed by a section called *Problems Plus*, which features examples of how to tackle challenging calculus problems. In selecting the Problems Plus problems we have kept in mind the following advice from David Hilbert: "A mathematical problem should be difficult in order to entice us, yet not inaccessible lest it mock our efforts." We have used these problems to great effect in our own calculus classes; it is gratifying to see how students respond to a challenge. James Stewart said, "When I put these challenging problems on assignments and tests I grade them in a different way . . . I reward a student significantly for ideas toward a solution and for recognizing which problem-solving principles are relevant."

Dual Treatment of Exponential and Logarithmic Functions

There are two possible ways of treating the exponential and logarithmic functions and each method has its passionate advocates. Because one often finds advocates of both approaches teaching the same course, we include full treatments of both methods. In Sections 6.2, 6.3, and 6.4 the exponential function is defined first, followed by the logarithmic function as its inverse. (Students have seen these functions introduced this way in previous courses.) In the alternative approach, presented in Sections 6.2*, 6.3*, and 6.4*, the logarithm is defined as an integral and the exponential function is its inverse. This latter method is, of course, less intuitive but more elegant. You can use whichever treatment you prefer.

If the first approach is taken, then much of Chapter 6 can be covered before Chapters 4 and 5, if desired. To accommodate this choice of presentation there are specially identified exercises involving integrals of exponential and logarithmic functions at the end of the appropriate sections of Chapters 4 and 5. This order of presentation allows a faster-paced course to teach the transcendental functions and the definite integral in the first semester of the course.

For instructors who would like to go even further in this direction an alternate edition of this book, called *Calculus: Early Transcendentals*, Ninth Edition, is available. In this version the exponential and logarithmic functions are introduced in the first chapter. Their limits and derivatives are found in the second and third chapters at the same time as polynomials and the other elementary functions.

Technology

When using technology, it is particularly important to clearly understand the concepts that underlie the images on the screen or the results of a calculation. When properly used, graphing calculators and computers are powerful tools for discovering and understanding those concepts. This textbook can be used either with or without technology—we use two special symbols to indicate clearly when a particular type of assistance from technology is required. The icon indicates an exercise that definitely requires the use of graphing software or a graphing calculator to aid in sketching a graph. (That is not to say that the technology can't be used on the other exercises as well.) The symbol T means that the assistance of software or a graphing calculator is needed beyond just graphing to complete the exercise. Freely available websites such as WolframAlpha.com or Symbolab.com are often suitable. In cases where the full resources of a computer algebra system, such as Maple or Mathematica, are needed, we state this in the exercise. Of course, technology doesn't make pencil and paper obsolete. Hand calculation and sketches are often preferable to technology for illustrating and reinforcing some concepts. Both instructors and students need to develop the ability to decide where using technology is appropriate and where more insight is gained by working out an exercise by hand.

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Stewart Website

Visit StewartCalculus.com for these additional materials:

- Homework Hints
- Solutions to the Concept Checks (from the review section of each chapter)
- Algebra and Analytic Geometry Review
- Lies My Calculator and Computer Told Me
- History of Mathematics, with links to recommended historical websites
- Additional Topics (complete with exercise sets): Fourier Series, Rotation of Axes, Formulas for the Remainder Theorem in Taylor Series
- Additional chapter on second-order differential equations, including the method of series solutions, and an appendix section reviewing complex numbers and complex exponential functions
- Instructor Area that includes archived problems (drill exercises that appeared in previous editions, together with their solutions)
- Challenge Problems (some from the Problems Plus sections from prior editions)
- Links, for particular topics, to outside Web resources

Content

Diagnostic Tests The book begins with four diagnostic tests, in Basic Algebra, Analytic Geometry, Functions, and Trigonometry.

A Preview of Calculus This is an overview of the subject and includes a list of questions to motivate the study of calculus.

1 Functions and Limits From the beginning, multiple representations of functions are stressed: verbal, numerical, visual, and algebraic. A discussion of mathematical models leads to a review of the standard functions from these four points of view. The material on limits is motivated by a prior discussion of the tangent and velocity problems. Limits are treated from descriptive, graphical, numerical, and algebraic points of view. Section 1.7, on the precise epsilon–delta defintion of a limit, is an optional section.

2 Derivatives The material on derivatives is covered in two sections in order to give students more time to get used to the idea of a derivative as a function. The examples and exercises explore the meanings of derivatives in various contexts. Higher derivatives are introduced in Section 2.2.

3 Applications of Differentiation The basic facts concerning extreme values and shapes of curves are deduced from the Mean Value Theorem. Graphing with technology emphasizes the interaction between calculus and machines and the analysis of families of curves. Some substantial optimization problems are provided, including an explanation of why you need to raise your head 42° to see the top of a rainbow.

4 Integrals The area problem and the distance problem serve to motivate the definite integral, with sigma notation introduced as needed. (Full coverage of sigma notation is provided in Appendix E.) Emphasis is placed on explaining the meanings of integrals in various contexts and on estimating their values from graphs and tables.

5 Applications of Integration This chapter presents the applications of integration—area, volume, work, average value—that can reasonably be done without specialized techniques of integration. General methods are emphasized. The goal is for students to be able to divide a quantity into small pieces, estimate with Riemann sums, and recognize the limit as an integral.

6 Inverse Functions: Exponential, Logarithmic, and Inverse Trigonometric Functions As discussed more fully on page xiv, only one of the two treatments of these functions need be covered. Exponential growth and decay are covered in this chapter.

7 Techniques of Integration All the standard methods are covered but, of course, the real challenge is to be able to recognize which technique is best used in a given situation. Accordingly, a strategy for evaluating integrals is explained in Section 7.5. The use of mathematical software is discussed in Section 7.6.

8 Further Applications of Integration This chapter contains the applications of integration—arc length and surface area—for which it is useful to have available all the techniques of integration, as well as applications to biology, economics, and physics (hydrostatic force and centers of mass). A section on probability is included. There are more applications here than can realistically be covered in a given course. Instructors may select applications suitable for their students and for which they themselves have enthusiasm.

9 Differential Equations Modeling is the theme that unifies this introductory treatment of differential equations. Direction fields and Euler's method are studied before separable and linear equations are solved explicitly, so that qualitative, numerical, and analytic approaches are given equal consideration. These methods are applied to the exponential, logistic, and other models for population growth. The first four or five sections of this chapter serve as a good introduction to first-order differential equations. An optional final section uses predator-prey models to illustrate systems of differential equations.

10 Parametric Equations and Polar Coordinates This chapter introduces parametric and polar curves and applies the methods of calculus to them. Parametric curves are well suited to projects that require graphing with technology; the two presented here involve families of curves and Bézier curves. A brief treatment of conic sections in polar coordinates prepares the way for Kepler's Laws in Chapter 13.

11 Sequences, Series, and Power Series The convergence tests have intuitive justifications (see Section 11.3) as well as formal proofs. Numerical estimates of sums of series are based on which test was used to prove convergence. The emphasis is on Taylor series and polynomials and their applications to physics.

12 Vectors and the Geometry of Space The material on three-dimensional analytic geometry and vectors is covered in this and the next chapter. Here we deal with vectors, the dot and cross products, lines, planes, and surfaces.

13 Vector Functions This chapter covers vector-valued functions, their derivatives and integrals, the length and curvature of space curves, and velocity and acceleration along space curves, culminating in Kepler's laws.

14 Partial Derivatives Functions of two or more variables are studied from verbal, numerical, visual, and algebraic points of view. In particular, partial derivatives are introduced by looking at a specific column in a table of values of the heat index (perceived air temperature) as a function of the actual temperature and the relative humidity.

15 Multiple Integrals Contour maps and the Midpoint Rule are used to estimate the average snowfall and average temperature in given regions. Double and triple integrals are used to compute volumes, surface areas, and (in projects) volumes of hyperspheres and volumes of intersections of three cylinders. Cylindrical and spherical coordinates are introduced in the context of evaluating triple integrals. Several applications are considered, including computing mass, charge, and probabilities.

16 Vector Calculus Vector fields are introduced through pictures of velocity fields showing San Francisco Bay wind patterns. The similarities among the Fundamental Theorem for line integrals, Green's Theorem, Stokes' Theorem, and the Divergence Theorem are emphasized.

17 Second-Order Differential Equations Since first-order differential equations are covered in Chapter 9, this online chapter deals with second-order linear differential equations, their application to vibrating springs and electric circuits, and series solutions.

Ancillaries

Calculus, Ninth Edition, is supported by a complete set of ancillaries. Each piece has been designed to enhance student understanding and to facilitate creative instruction.

Ancillaries for Instructors

Instructor's Guide
by Douglas Shaw

Each section of the text is discussed from several viewpoints. Available online at the Instructor's Companion Site, the Instructor's Guide *contains suggested time to allot, points to stress, text discussion topics, core materials for lecture, workshop/discussion suggestions, group work exercises in a form suitable for handout, and suggested homework assignments.*

Complete Solutions Manual

Single Variable Calculus, Ninth Edition
Chapters 1–11
By Joshua Babbin, Scott Barnett, and Jeffery A. Cole

Multivariable Calculus, Ninth Edition
Chapters 10–16
By Joshua Babbin and Gina Sanders

Includes worked-out solutions to all exercises in the text. Both volumes of the Complete Solutions Manual *are available online at the Instructor's Companion Site.*

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Contains text-specific multiple-choice and free response test items and is available online at the Instructor's Companion Site.

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Ancillaries for Students

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Acknowledgments

One of the main factors aiding in the preparation of this edition is the cogent advice from a large number of reviewers, all of whom have extensive experience teaching calculus. We greatly appreciate their suggestions and the time they spent to understand the approach taken in this book. We have learned something from each of them.

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Theodore W. Wilcox, *Rochester Institute of Technology*
Steven Willard, *University of Alberta*
David Williams, *Clayton State University*
Robert Wilson, *University of Wisconsin–Madison*
Jerome Wolbert, *University of Michigan–Ann Arbor*
Dennis H. Wortman, *University of Massachusetts, Boston*
Mary Wright, *Southern Illinois University–Carbondale*
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We are deeply indebted to our friend and colleague Lothar Redlin who began working with us on this revision shortly before his untimely death in 2018. Lothar's deep insights into mathematics and its pedagogy, and his lightning fast problem-solving skills, were invaluable assets.

We especially thank Kathi Townes of TECHarts, our production service and copyeditor (for this as well as the past several editions). Her extraordinary ability to recall any detail of the manuscript as needed, her facility in simultaneously handling different editing tasks, and her comprehensive familiarity with the book were key factors in its accuracy and timely production. We also thank Lori Heckelman for the elegant and precise rendering of the new illustrations.

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This textbook has benefited greatly over the past three decades from the advice and guidance of some of the best mathematics editors: Ron Munro, Harry Campbell, Craig Barth, Jeremy Hayhurst, Gary Ostedt, Bob Pirtle, Richard Stratton, Liz Covello, Neha Taleja, and now Gary Whalen. They have all contributed significantly to the success of this book. Prominently, Gary Whalen's broad knowledge of current issues in the teaching of mathematics and his continual research into creating better ways of using technology as a teaching and learning tool were invaluable resources in the creation of this edition.

JAMES STEWART
DANIEL CLEGG
SALEEM WATSON

A Tribute to James Stewart

JAMES STEWART had a singular gift for teaching mathematics. The large lecture halls where he taught his calculus classes were always packed to capacity with students, whom he held engaged with interest and anticipation as he led them to discover a new concept or the solution to a stimulating problem. Stewart presented calculus the way he viewed it—as a rich subject with intuitive concepts, wonderful problems, powerful applications, and a fascinating history. As a testament to his success in teaching and lecturing, many of his students went on to become mathematicians, scientists, and engineers—and more than a few are now university professors themselves. It was his students who first suggested that he write a calculus textbook of his own. Over the years, former students, by then working scientists and engineers, would call him to discuss mathematical problems that they encountered in their work; some of these discussions resulted in new exercises or projects in the book.

We each met James Stewart—or Jim as he liked us to call him—through his teaching and lecturing, resulting in his inviting us to coauthor mathematics textbooks with him. In the years we have known him, he was in turn our teacher, mentor, and friend.

Jim had several special talents whose combination perhaps uniquely qualified him to write such a beautiful calculus textbook—a textbook with a narrative that speaks to students and that combines the fundamentals of calculus with conceptual insights on how to think about them. Jim always listened carefully to his students in order to find out precisely where they may have had difficulty with a concept. Crucially, Jim really enjoyed hard work—a necessary trait for completing the immense task of writing a calculus book. As his coauthors, we enjoyed his contagious enthusiasm and optimism, making the time we spent with him always fun and productive, never stressful.

Most would agree that writing a calculus textbook is a major enough feat for one lifetime, but amazingly, Jim had many other interests and accomplishments: he played violin professionally in the Hamilton and McMaster Philharmonic Orchestras for many years, he had an enduring passion for architecture, he was a patron of the arts and cared deeply about many social and humanitarian causes. He was also a world traveler, an eclectic art collector, and even a gourmet cook.

James Stewart was an extraordinary person, mathematician, and teacher. It has been our honor and privilege to be his coauthors and friends.

DANIEL CLEGG

SALEEM WATSON

About the Authors

For more than two decades, Daniel Clegg and Saleem Watson have worked with James Stewart on writing mathematics textbooks. The close working relationship between them was particularly productive because they shared a common viewpoint on teaching mathematics and on writing mathematics. In a 2014 interview James Stewart remarked on their collaborations: "We discovered that we could think in the same way . . . we agreed on almost everything, which is kind of rare."

Daniel Clegg and Saleem Watson met James Stewart in different ways, yet in each case their initial encounter turned out to be the beginning of a long association. Stewart spotted Daniel's talent for teaching during a chance meeting at a mathematics conference and asked him to review the manuscript for an upcoming edition of *Calculus* and to author the multivariable solutions manual. Since that time Daniel has played an ever-increasing role in the making of several editions of the Stewart calculus books. He and Stewart have also coauthored an applied calculus textbook. Stewart first met Saleem when Saleem was a student in his graduate mathematics class. Later Stewart spent a sabbatical leave doing research with Saleem at Penn State University, where Saleem was an instructor at the time. Stewart asked Saleem and Lothar Redlin (also a student of Stewart's) to join him in writing a series of precalculus textbooks; their many years of collaboration resulted in several editions of these books.

JAMES STEWART was professor of mathematics at McMaster University and the University of Toronto for many years. James did graduate studies at Stanford University and the University of Toronto, and subsequently did research at the University of London. His research field was Harmonic Analysis and he also studied the connections between mathematics and music.

DANIEL CLEGG is professor of mathematics at Palomar College in Southern California. He did undergraduate studies at California State University, Fullerton and graduate studies at the University of California, Los Angeles (UCLA). Daniel is a consummate teacher; he has been teaching mathematics ever since he was a graduate student at UCLA.

SALEEM WATSON is professor emeritus of mathematics at California State University, Long Beach. He did undergraduate studies at Andrews University in Michigan and graduate studies at Dalhousie University and McMaster University. After completing a research fellowship at the University of Warsaw, he taught for several years at Penn State before joining the mathematics department at California State University, Long Beach.

Stewart and Clegg have published *Brief Applied Calculus.*

Stewart, Redlin, and Watson have published *Precalculus: Mathematics for Calculus, College Algebra, Trigonometry, Algebra and Trigonometry,* and (with Phyllis Panman) *College Algebra: Concepts and Contexts.*

Technology in the Ninth Edition

Graphing and computing devices are valuable tools for learning and exploring calculus, and some have become well established in calculus instruction. Graphing calculators are useful for drawing graphs and performing some numerical calculations, like approximating solutions to equations or numerically evaluating derivatives (Chapter 2) or definite integrals (Chapter 4). Mathematical software packages called computer algebra systems (CAS, for short) are more powerful tools. Despite the name, algebra represents only a small subset of the capabilities of a CAS. In particular, a CAS can do mathematics symbolically rather than just numerically. It can find exact solutions to equations and exact formulas for derivatives and integrals.

We now have access to a wider variety of tools of varying capabilities than ever before. These include Web-based resources (some of which are free of charge) and apps for smartphones and tablets. Many of these resources include at least some CAS functionality, so some exercises that may have typically required a CAS can now be completed using these alternate tools.

In this edition, rather than refer to a specific type of device (a graphing calculator, for instance) or software package (such as a CAS), we indicate the type of capability that is needed to work an exercise.

Graphing Icon

The appearance of this icon beside an exercise indicates that you are expected to use a machine or software to help you draw the graph. In many cases, a graphing calculator will suffice. Websites such as Desmos.com provide similar capability. If the graph is in 3D (see Chapters 12–16), WolframAlpha.com is a good resource. There are also many graphing software applications for computers, smartphones, and tablets. If an exercise asks for a graph but no graphing icon is shown, then you are expected to draw the graph by hand. In Chapter 1 we review graphs of basic functions and discuss how to use transformations to graph modified versions of these basic functions.

T Technology Icon

This icon is used to indicate that software or a device with abilities beyond just graphing is needed to complete the exercise. Many graphing calculators and software resources can provide numerical approximations when needed. For working with mathematics symbolically, websites like WolframAlpha.com or Symbolab.com are helpful, as are more advanced graphing calculators such as the Texas Instrument TI-89 or TI-Nspire CAS. If the full power of a CAS is needed, this will be stated in the exercise, and access to software packages such as Mathematica, Maple, MATLAB, or SageMath may be required. If an exercise does not include a technology icon, then you are expected to evaluate limits, derivatives, and integrals, or solve equations by hand, arriving at exact answers. No technology is needed for these exercises beyond perhaps a basic scientific calculator.

To the Student

Reading a calculus textbook is different from reading a story or a news article. Don't be discouraged if you have to read a passage more than once in order to understand it. You should have pencil and paper and calculator at hand to sketch a diagram or make a calculation.

Some students start by trying their homework problems and read the text only if they get stuck on an exercise. We suggest that a far better plan is to read and understand a section of the text before attempting the exercises. In particular, you should look at the definitions to see the exact meanings of the terms. And before you read each example, we suggest that you cover up the solution and try solving the problem yourself.

Part of the aim of this course is to train you to think logically. Learn to write the solutions of the exercises in a connected, step-by-step fashion with explanatory sentences—not just a string of disconnected equations or formulas.

The answers to the odd-numbered exercises appear at the back of the book, in Appendix G. Some exercises ask for a verbal explanation or interpretation or description. In such cases there is no single correct way of expressing the answer, so don't worry that you haven't found the definitive answer. In addition, there are often several different forms in which to express a numerical or algebraic answer, so if your answer differs from the given one, don't immediately assume you're wrong. For example, if the answer given in the back of the book is $\sqrt{2} - 1$ and you obtain $1/(1 + \sqrt{2})$, then you're correct and rationalizing the denominator will show that the answers are equivalent.

The icon indicates an exercise that definitely requires the use of either a graphing calculator or a computer with graphing software to help you sketch the graph. But that doesn't mean that graphing devices can't be used to check your work on the other exercises as well. The symbol T indicates that technological assistance beyond just graphing is needed to complete the exercise. (See Technology in the Ninth Edition for more details.)

You will also encounter the symbol, which warns you against committing an error. This symbol is placed in the margin in situations where many students tend to make the same mistake.

Homework Hints are available for many exercises. These hints can be found on StewartCalculus.com as well as in WebAssign. The homework hints ask you questions that allow you to make progress toward a solution without actually giving you the answer. If a particular hint doesn't enable you to solve the problem, you can click to reveal the next hint.

We recommend that you keep this book for reference purposes after you finish the course. Because you will likely forget some of the specific details of calculus, the book will serve as a useful reminder when you need to use calculus in subsequent courses. And, because this book contains more material than can be covered in any one course, it can also serve as a valuable resource for a working scientist or engineer.

Calculus is an exciting subject, justly considered to be one of the greatest achievements of the human intellect. We hope you will discover that it is not only useful but also intrinsically beautiful.

Diagnostic Tests

Success in calculus depends to a large extent on knowledge of the mathematics that precedes calculus: algebra, analytic geometry, functions, and trigonometry. The following tests are intended to diagnose weaknesses that you might have in these areas. After taking each test you can check your answers against the given answers and, if necessary, refresh your skills by referring to the review materials that are provided.

A Diagnostic Test: Algebra

1. Evaluate each expression without using a calculator.

(a) $(-3)^4$ (b) -3^4 (c) 3^{-4}

(d) $\dfrac{5^{23}}{5^{21}}$ (e) $\left(\dfrac{2}{3}\right)^{-2}$ (f) $16^{-3/4}$

2. Simplify each expression. Write your answer without negative exponents.

(a) $\sqrt{200} - \sqrt{32}$

(b) $(3a^3b^3)(4ab^2)^2$

(c) $\left(\dfrac{3x^{3/2}y^3}{x^2y^{-1/2}}\right)^{-2}$

3. Expand and simplify.

(a) $3(x + 6) + 4(2x - 5)$ (b) $(x + 3)(4x - 5)$

(c) $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b})$ (d) $(2x + 3)^2$

(e) $(x + 2)^3$

4. Factor each expression.

(a) $4x^2 - 25$ (b) $2x^2 + 5x - 12$

(c) $x^3 - 3x^2 - 4x + 12$ (d) $x^4 + 27x$

(e) $3x^{3/2} - 9x^{1/2} + 6x^{-1/2}$ (f) $x^3y - 4xy$

5. Simplify the rational expression.

(a) $\dfrac{x^2 + 3x + 2}{x^2 - x - 2}$ (b) $\dfrac{2x^2 - x - 1}{x^2 - 9} \cdot \dfrac{x + 3}{2x + 1}$

(c) $\dfrac{x^2}{x^2 - 4} - \dfrac{x + 1}{x + 2}$ (d) $\dfrac{\dfrac{y}{x} - \dfrac{x}{y}}{\dfrac{1}{y} - \dfrac{1}{x}}$

6. Rationalize the expression and simplify.

(a) $\dfrac{\sqrt{10}}{\sqrt{5} - 2}$ (b) $\dfrac{\sqrt{4 + h} - 2}{h}$

7. Rewrite by completing the square.

(a) $x^2 + x + 1$ (b) $2x^2 - 12x + 11$

8. Solve the equation. (Find only the real solutions.)

(a) $x + 5 = 14 - \frac{1}{2}x$ (b) $\dfrac{2x}{x + 1} = \dfrac{2x - 1}{x}$

(c) $x^2 - x - 12 = 0$ (d) $2x^2 + 4x + 1 = 0$

(e) $x^4 - 3x^2 + 2 = 0$ (f) $3|x - 4| = 10$

(g) $2x(4 - x)^{-1/2} - 3\sqrt{4 - x} = 0$

9. Solve each inequality. Write your answer using interval notation.

(a) $-4 < 5 - 3x \leqslant 17$ (b) $x^2 < 2x + 8$

(c) $x(x - 1)(x + 2) > 0$ (d) $|x - 4| < 3$

(e) $\dfrac{2x - 3}{x + 1} \leqslant 1$

10. State whether each equation is true or false.

(a) $(p + q)^2 = p^2 + q^2$ (b) $\sqrt{ab} = \sqrt{a}\,\sqrt{b}$

(c) $\sqrt{a^2 + b^2} = a + b$ (d) $\dfrac{1 + TC}{C} = 1 + T$

(e) $\dfrac{1}{x - y} = \dfrac{1}{x} - \dfrac{1}{y}$ (f) $\dfrac{1/x}{a/x - b/x} = \dfrac{1}{a - b}$

ANSWERS TO DIAGNOSTIC TEST A: ALGEBRA

1. (a) 81 (b) -81 (c) $\frac{1}{81}$
(d) 25 (e) $\frac{9}{4}$ (f) $\frac{1}{8}$

2. (a) $6\sqrt{2}$ (b) $48a^5b^7$ (c) $\dfrac{x}{9y^7}$

3. (a) $11x - 2$ (b) $4x^2 + 7x - 15$
(c) $a - b$ (d) $4x^2 + 12x + 9$
(e) $x^3 + 6x^2 + 12x + 8$

4. (a) $(2x - 5)(2x + 5)$ (b) $(2x - 3)(x + 4)$
(c) $(x - 3)(x - 2)(x + 2)$ (d) $x(x + 3)(x^2 - 3x + 9)$
(e) $3x^{-1/2}(x - 1)(x - 2)$ (f) $xy(x - 2)(x + 2)$

5. (a) $\dfrac{x + 2}{x - 2}$ (b) $\dfrac{x - 1}{x - 3}$
(c) $\dfrac{1}{x - 2}$ (d) $-(x + y)$

6. (a) $5\sqrt{2} + 2\sqrt{10}$ (b) $\dfrac{1}{\sqrt{4 + h} + 2}$

7. (a) $\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$ (b) $2(x - 3)^2 - 7$

8. (a) 6 (b) 1 (c) $-3, 4$
(d) $-1 \pm \frac{1}{2}\sqrt{2}$ (e) $\pm 1, \pm\sqrt{2}$ (f) $\frac{2}{3}, \frac{22}{3}$
(g) $\frac{12}{5}$

9. (a) $[-4, 3)$ (b) $(-2, 4)$
(c) $(-2, 0) \cup (1, \infty)$ (d) $(1, 7)$
(e) $(-1, 4]$

10. (a) False (b) True (c) False
(d) False (e) False (f) True

If you had difficulty with these problems, you may wish to consult the Review of Algebra on the website **StewartCalculus.com.**

B | Diagnostic Test: Analytic Geometry

1. Find an equation for the line that passes through the point $(2, -5)$ and
(a) has slope -3
(b) is parallel to the x-axis
(c) is parallel to the y-axis
(d) is parallel to the line $2x - 4y = 3$

2. Find an equation for the circle that has center $(-1, 4)$ and passes through the point $(3, -2)$.

3. Find the center and radius of the circle with equation $x^2 + y^2 - 6x + 10y + 9 = 0$.

4. Let $A(-7, 4)$ and $B(5, -12)$ be points in the plane.
(a) Find the slope of the line that contains A and B.
(b) Find an equation of the line that passes through A and B. What are the intercepts?
(c) Find the midpoint of the segment AB.
(d) Find the length of the segment AB.
(e) Find an equation of the perpendicular bisector of AB.
(f) Find an equation of the circle for which AB is a diameter.

5. Sketch the region in the xy-plane defined by the equation or inequalities.
(a) $-1 \leqslant y \leqslant 3$
(b) $|x| < 4$ and $|y| < 2$
(c) $y < 1 - \frac{1}{2}x$
(d) $y \geqslant x^2 - 1$
(e) $x^2 + y^2 < 4$
(f) $9x^2 + 16y^2 = 144$

ANSWERS TO DIAGNOSTIC TEST B: ANALYTIC GEOMETRY

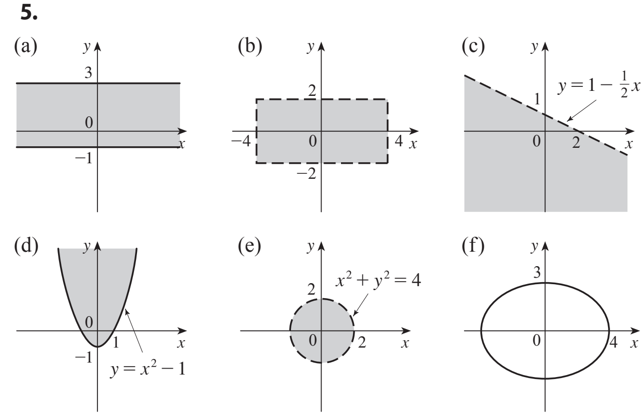
1. (a) $y = -3x + 1$ (b) $y = -5$
(c) $x = 2$ (d) $y = \frac{1}{2}x - 6$

2. $(x + 1)^2 + (y - 4)^2 = 52$

3. Center $(3, -5)$, radius 5

4. (a) $-\frac{4}{3}$
(b) $4x + 3y + 16 = 0$; x-intercept -4, y-intercept $-\frac{16}{3}$
(c) $(-1, -4)$
(d) 20
(e) $3x - 4y = 13$
(f) $(x + 1)^2 + (y + 4)^2 = 100$

5.



If you had difficulty with these problems, you may wish to consult the review of analytic geometry in Appendixes B and C.

C | Diagnostic Test: Functions

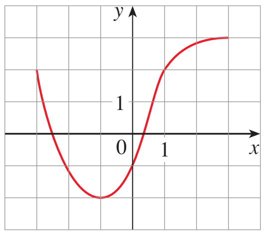


FIGURE FOR PROBLEM 1

1. The graph of a function f is given at the left.
(a) State the value of $f(-1)$.
(b) Estimate the value of $f(2)$.
(c) For what values of x is $f(x) = 2$?
(d) Estimate the values of x such that $f(x) = 0$.
(e) State the domain and range of f.

2. If $f(x) = x^3$, evaluate the difference quotient $\dfrac{f(2+h) - f(2)}{h}$ and simplify your answer.

3. Find the domain of the function.
(a) $f(x) = \dfrac{2x+1}{x^2+x-2}$ (b) $g(x) = \dfrac{\sqrt[3]{x}}{x^2+1}$ (c) $h(x) = \sqrt{4-x} + \sqrt{x^2-1}$

4. How are graphs of the functions obtained from the graph of f?
(a) $y = -f(x)$ (b) $y = 2f(x) - 1$ (c) $y = f(x-3) + 2$

5. Without using a calculator, make a rough sketch of the graph.
(a) $y = x^3$ (b) $y = (x+1)^3$ (c) $y = (x-2)^3 + 3$
(d) $y = 4 - x^2$ (e) $y = \sqrt{x}$ (f) $y = 2\sqrt{x}$
(g) $y = -2^x$ (h) $y = 1 + x^{-1}$

6. Let $f(x) = \begin{cases} 1 - x^2 & \text{if } x \leqslant 0 \\ 2x + 1 & \text{if } x > 0 \end{cases}$
(a) Evaluate $f(-2)$ and $f(1)$. (b) Sketch the graph of f.

7. If $f(x) = x^2 + 2x - 1$ and $g(x) = 2x - 3$, find each of the following functions.
(a) $f \circ g$ (b) $g \circ f$ (c) $g \circ g \circ g$

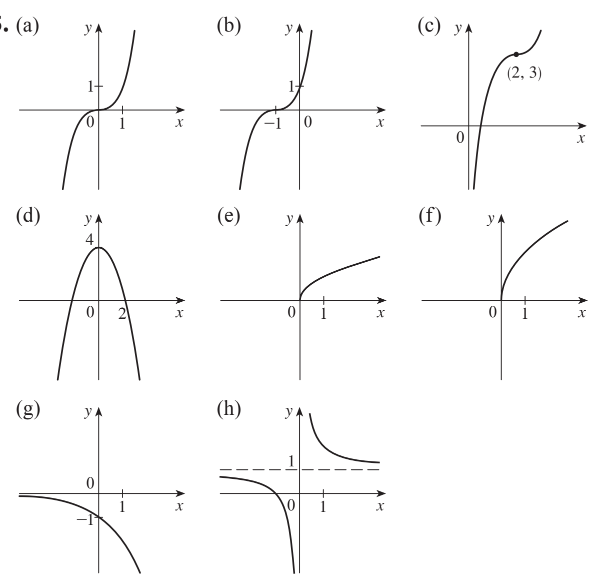
ANSWERS TO DIAGNOSTIC TEST C: FUNCTIONS

1. (a) -2 (b) 2.8
(c) $-3, 1$ (d) $-2.5, 0.3$
(e) $[-3, 3], [-2, 3]$

2. $12 + 6h + h^2$

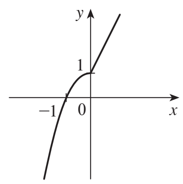
3. (a) $(-\infty, -2) \cup (-2, 1) \cup (1, \infty)$
(b) $(-\infty, \infty)$
(c) $(-\infty, -1] \cup [1, 4]$

4. (a) Reflect about the x-axis
(b) Stretch vertically by a factor of 2, then shift 1 unit downward
(c) Shift 3 units to the right and 2 units upward

5.


6. (a) $-3, 3$

(b)



7. (a) $(f \circ g)(x) = 4x^2 - 8x + 2$
(b) $(g \circ f)(x) = 2x^2 + 4x - 5$
(c) $(g \circ g \circ g)(x) = 8x - 21$

If you had difficulty with these problems, you should look at sections 1.1–1.3 of this book.

D Diagnostic Test: Trigonometry

1. Convert from degrees to radians.
(a) $300°$ (b) $-18°$

2. Convert from radians to degrees.
(a) $5\pi/6$ (b) 2

3. Find the length of an arc of a circle with radius 12 cm if the arc subtends a central angle of 30°.

4. Find the exact values.
(a) $\tan(\pi/3)$ (b) $\sin(7\pi/6)$ (c) $\sec(5\pi/3)$

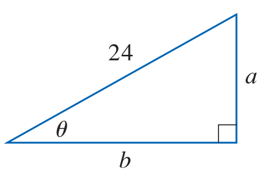


FIGURE FOR PROBLEM 5

5. Express the lengths a and b in the figure in terms of θ.

6. If $\sin x = \frac{1}{3}$ and $\sec y = \frac{5}{4}$, where x and y lie between 0 and $\pi/2$, evaluate $\sin(x + y)$.

7. Prove the identities.
(a) $\tan\theta \sin\theta + \cos\theta = \sec\theta$ (b) $\dfrac{2\tan x}{1 + \tan^2 x} = \sin 2x$

8. Find all values of x such that $\sin 2x = \sin x$ and $0 \leqslant x \leqslant 2\pi$.

9. Sketch the graph of the function $y = 1 + \sin 2x$ without using a calculator.

ANSWERS TO DIAGNOSTIC TEST D: TRIGONOMETRY

1. (a) $5\pi/3$ (b) $-\pi/10$

2. (a) $150°$ (b) $360°/\pi \approx 114.6°$

3. 2π cm

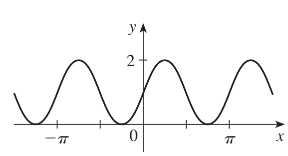
4. (a) $\sqrt{3}$ (b) $-\frac{1}{2}$ (c) 2

5. $a = 24\sin\theta, b = 24\cos\theta$

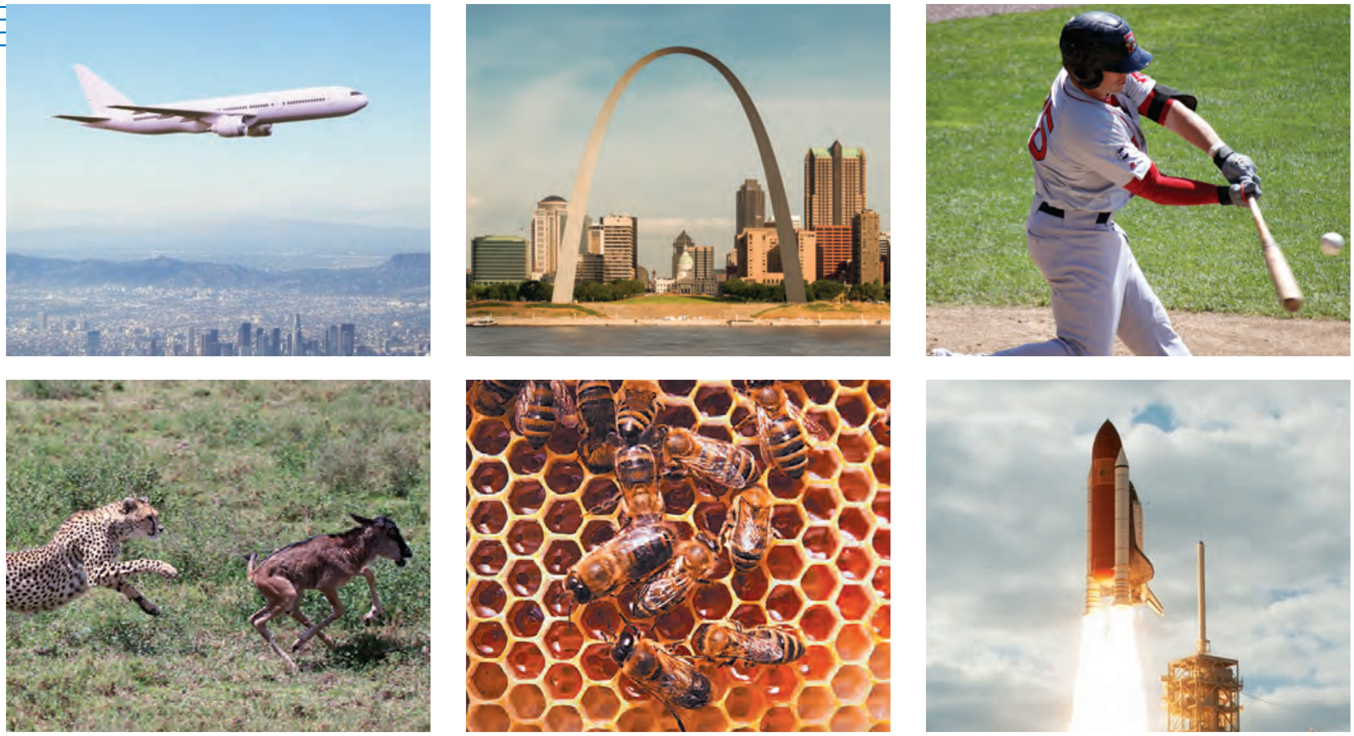
6. $\frac{1}{15}(4 + 6\sqrt{2})$

8. $0, \pi/3, \pi, 5\pi/3, 2\pi$

9.



If you had difficulty with these problems, you should look at Appendix D of this book.



By the time you finish this ourse, you will be able to determine where a pilot should start descent for a smooth landing, find the length of the cu ve used to design the Gateway Arch in St. Louis, compute the force on a baseball bat when it strikes the ball, predict the population sizes for competing predator-prey species, show that bees form the cells of a beehive in a way that uses the least amount of wax, and estimate the amount of fuel needed to propel a rocket into orbit.

Top row: Who is Danny / Shutterstock.com; iStock.com/gnagel; Richard Paul Kane / Shutterstock

Bottom row: Bruce Ellis / Shutterstock.com; Kostiantyn Kravchenko / Shutterstock.com; Ben Cooper / Science Faction / Getty Images

A Preview of Calculus

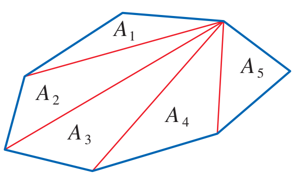
CALCULUS IS FUNDAMENTALLY DIFFERENT from the mathematics that you have studied previously: calculus is less static and more dynamic. It is concerned with change and motion; it deals with quantities that approach other quantities. For that reason it may be useful to have an overview of calculus before beginning your study of the subject. Here we give a preview of some of the main ideas of calculus and show how their foundations are built upon the concept of a *limit*.

What Is Calculus?

The world around us is continually changing—populations increase, a cup of coffee cools, a stone falls, chemicals react with one another, currency values fluctuate, and so on. We would like to be able to analyze quantities or processes that are undergoing continuous change. For example, if a stone falls 10 feet each second we could easily tell how fast it is falling at any time, but this is *not* what happens—the stone falls faster and faster, its speed changing at each instant. In studying calculus, we will learn how to model (or describe) such instantaneously changing processes and how to find the cumulative effect of these changes.

Calculus builds on what you have learned in algebra and analytic geometry but advances these ideas spectacularly. Its uses extend to nearly every field of human activity. You will encounter numerous applications of calculus throughout this book.

At its core, calculus revolves around two key problems involving the graphs of functions—the *area problem* and the *tangent problem*—and an unexpected relationship between them. Solving these problems is useful because the area under the graph of a function and the tangent to the graph of a function have many important interpretations in a variety of contexts.



$A = A_1 + A_2 + A_3 + A_4 + A_5$

FIGURE 1

The Area Problem

The origins of calculus go back at least 2500 years to the ancient Greeks, who found areas using the "method of exhaustion." They knew how to find the area A of any polygon by dividing it into triangles, as in Figure 1, and adding the areas of these triangles.

It is a much more difficult problem to find the area of a curved figure. The Greek method of exhaustion was to inscribe polygons in the figure and circumscribe polygons about the figure, and then let the number of sides of the polygons increase. Figure 2 illustrates this process for the special case of a circle with inscribed regular polygons.

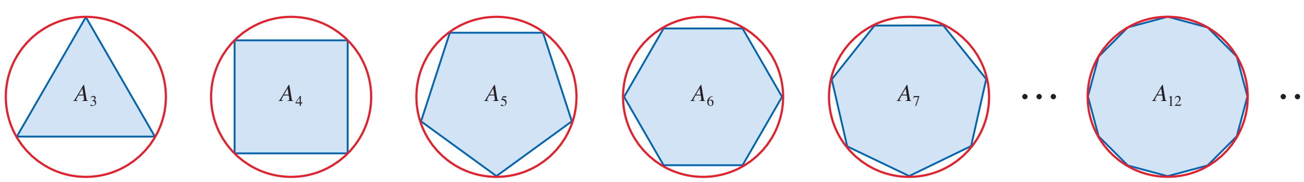


FIGURE 2

Let A_n be the area of the inscribed regular polygon of n sides. As n increases, it appears that A_n gets closer and closer to the area of the circle. We say that the area A of the circle is the *limit* of the areas of the inscribed polygons, and we write

$$A = \lim_{n \to \infty} A_n$$

The Greeks themselves did not use limits explicitly. However, by indirect reasoning, Eudoxus (fifth century BC) used exhaustion to prove the familiar formula for the area of a circle: $A = \pi r^2$.

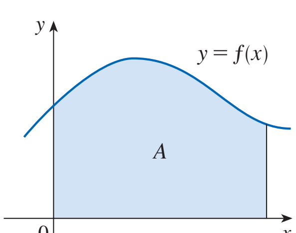


FIGURE 3
The area A of the region under the graph of f

We will use a similar idea in Chapter 4 to find areas of regions of the type shown in Figure 3. We approximate such an area by areas of rectangles as shown in Figure 4. If we approximate the area A of the region under the graph of f by using n rectangles $R_1, R_2, \ldots, R_n$, then the approximate area is

$$A_n = R_1 + R_2 + \cdots + R_n$$

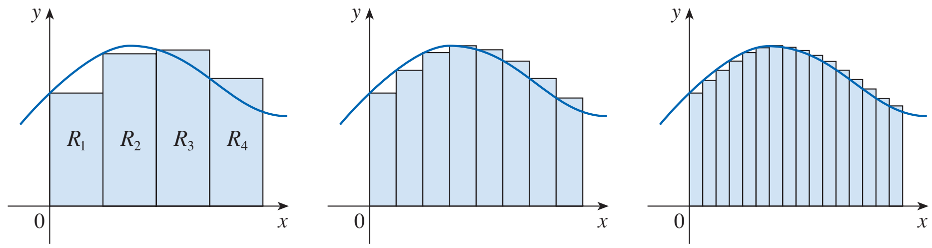


FIGURE 4 Approximating the area A using rectangles

Now imagine that we increase the number of rectangles (as the width of each one decreases) and calculate A as the limit of these sums of areas of rectangles:

$$A = \lim_{n \to \infty} A_n$$

In Chapter 4 we will learn how to calculate such limits.

The area problem is the central problem in the branch of calculus called *integral calculus*; it is important because the area under the graph of a function has different interpretations depending on what the function represents. In fact, the techniques that we develop for finding areas will also enable us to compute the volume of a solid, the length of a curve, the force of water against a dam, the mass and center of mass of a rod, the work done in pumping water out of a tank, and the amount of fuel needed to send a rocket into orbit.

The Tangent Problem

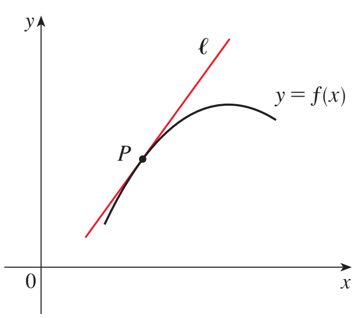


FIGURE 5
The tangent line at P

Consider the problem of trying to find an equation of the tangent line ℓ to a curve with equation $y = f(x)$ at a given point P. (We will give a precise definition of a tangent line in Chapter 1; for now you can think of it as the line that touches the curve at P and follows the direction of the curve at P, as in Figure 5.) Because the point P lies on the tangent line, we can find the equation of ℓ if we know its slope m. The problem is that we need two points to compute the slope and we know only one point, P, on ℓ. To get around the problem we first find an approximation to m by taking a nearby point Q on the curve and computing the slope m_{PQ} of the secant line PQ.

Now imagine that Q moves along the curve toward P as in Figure 6. You can see that the secant line PQ rotates and approaches the tangent line ℓ as its limiting position. This

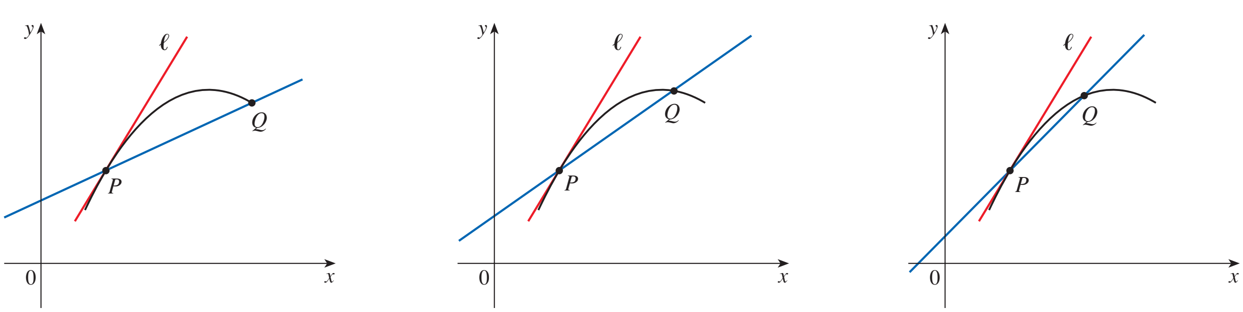


FIGURE 6 The secant lines approach the tangent line as Q approaches P.

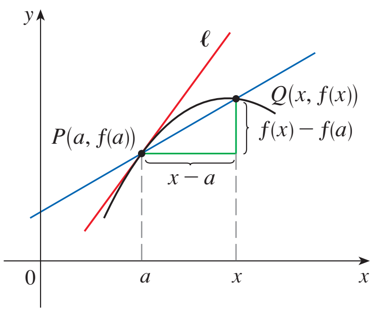


FIGURE 7
The secant line *PQ*

means that the slope m_{PQ} of the secant line becomes closer and closer to the slope m of the tangent line. We write

$$m = \lim_{Q \to P} m_{PQ}$$

and say that m is the limit of m_{PQ} as Q approaches P along the curve.

Notice from Figure 7 that if P is the point $(a, f(a))$ and Q is the point $(x, f(x))$, then

$$m_{PQ} = \frac{f(x) - f(a)}{x - a}$$

Because x approaches a as Q approaches P, an equivalent expression for the slope of the tangent line is

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

In Chapter 1 we will learn rules for calculating such limits.

The tangent problem has given rise to the branch of calculus called *differential calculus*; it is important because the slope of a tangent to the graph of a function can have different interpretations depending on the context. For instance, solving the tangent problem allows us to find the instantaneous speed of a falling stone, the rate of change of a chemical reaction, or the direction of the forces on a hanging chain.

A Relationship between the Area and Tangent Problems

The area and tangent problems seem to be very different problems but, surprisingly, the problems are closely related—in fact, they are so closely related that solving one of them leads to a solution of the other. The relationship between these two problems is introduced in Chapter 4; it is the central discovery in calculus and is appropriately named the Fundamental Theorem of Calculus. Perhaps most importantly, the Fundamental Theorem vastly simplifies the solution of the area problem, making it possible to find areas without having to approximate by rectangles and evaluate the associated limits.

Isaac Newton (1642–1727) and Gottfried Leibniz (1646–1716) are credited with the invention of calculus because they were the first to recognize the importance of the Fundamental Theorem of Calculus and to utilize it as a tool for solving real-world problems. In studying calculus you will discover these powerful results for yourself.

Summary

We have seen that the concept of a limit arises in finding the area of a region and in finding the slope of a tangent line to a curve. It is this basic idea of a limit that sets calculus apart from other areas of mathematics. In fact, we could define calculus as the part of mathematics that deals with limits. We have mentioned that areas under curves and slopes of tangent lines to curves have many different interpretations in a variety of contexts. Finally, we have discussed that the area and tangent problems are closely related.

After Isaac Newton invented his version of calculus, he used it to explain the motion of the planets around the sun, giving a definitive answer to a centuries-long quest for a description of our solar system. Today calculus is applied in a great variety of contexts, such as determining the orbits of satellites and spacecraft, predicting population sizes, forecasting weather, measuring cardiac output, and gauging the efficiency of an economic market.

In order to convey a sense of the power and versatility of calculus, we conclude with a list of some of the questions that you will be able to answer using calculus.

1. How can we design a roller coaster for a safe and smooth ride?
(See the Applied Project following Section 2.3.)

2. How far away from an airport should a pilot start descent?
(See the Applied Project following Section 2.5.)

3. How can we explain the fact that the angle of elevation from an observer up to the highest point in a rainbow is always 42°?
(See the Applied Project following Section 3.1.)

4. How can we estimate the amount of work that was required to build the Pyramid of Khufu in ancient Egypt?
(See Exercise 36 in Section 5.4.)

5. With what speed must a projectile be launched so that it escapes the earth's gravitation pull?
(See Exercise 77 in Section 7.8.)

6. How can we explain the changes in the thickness of sea ice over time and why cracks in the ice tend to "heal"?
(See Exercise 56 in Section 9.3.)

7. Does a ball thrown upward take longer to reach its maximum height or to fall back down to its original height?
(See the Applied Project following Section 9.5.)

8. How can we fit curves together to design shapes to represent letters on a laser printer?
(See the Applied Project following Section 10.2.)

9. How can we explain the fact that planets and satellites move in elliptical orbits?
(See the Applied Project following Section 13.4.)

10. How can we distribute water flow among turbines at a hydroelectric station so as to maximize the total energy production?
(See the Applied Project following Section 14.8.)

The electrical power produced by a wind turbine can be estimated by a mathematical function that incorporates several factors. We will explore this function in Exercise 1.2.25 and determine the expected power output of a particular turbine for various wind speeds.

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1 Functions and Limits

THE FUNDAMENTAL OBJECTS THAT WE deal with in calculus are functions. We stress that a function can be represented in different ways: by an equation, in a table, by a graph, or in words. We look at the main types of functions that occur in calculus and describe the process of using these functions as mathematical models of real-world phenomena.

In *A Preview of Calculus* (immediately preceding this chapter) we saw how the idea of a limit underlies the various branches of calculus. It is therefore appropriate to begin our study of calculus by investigating limits of functions and their properties.

1.1 Four Ways to Represent a Function

Functions

Functions arise whenever one quantity depends on another. Consider the following four situations.

A. The area A of a circle depends on the radius r of the circle. The rule that connects r and A is given by the equation $A = \pi r^2$. With each positive number r there is associated one value of A, and we say that A is a *function* of r.

B. The human population of the world P depends on the time t. Table 1 gives estimates of the world population P at time t, for certain years. For instance,

$$P \approx 2{,}560{,}000{,}000 \qquad \text{when } t = 1950$$

For each value of the time t there is a corresponding value of P, and we say that P is a function of t.

Table 1 World Population

Year	Population (millions)
1900	1650
1910	1750
1920	1860
1930	2070
1940	2300
1950	2560
1960	3040
1970	3710
1980	4450
1990	5280
2000	6080
2010	6870

C. The cost C of mailing an envelope depends on its weight w. Although there is no simple formula that connects w and C, the post office has a rule for determining C when w is known.

D. The vertical acceleration a of the ground as measured by a seismograph during an earthquake is a function of the elapsed time t. Figure 1 shows a graph generated by seismic activity during the Northridge earthquake that shook Los Angeles in 1994. For a given value of t, the graph provides a corresponding value of a.

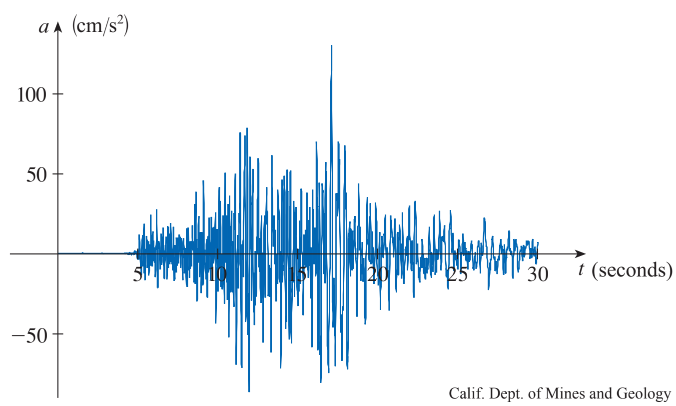

FIGURE 1
Vertical ground acceleration during the Northridge earthquake

Each of these examples describes a rule whereby, given a number (r in Example A), another number (A) is assigned. In each case we say that the second number is a function of the first number. If f represents the rule that connects A to r in Example A, then we express this in **function notation** as $A = f(r)$.

> A **function** f is a rule that assigns to each element x in a set D exactly one element, called $f(x)$, in a set E.

We usually consider functions for which the sets D and E are sets of real numbers. The set D is called the **domain** of the function. The number $f(x)$ is the **value of f at x** and is read "f of x." The **range** of f is the set of all possible values of $f(x)$ as x varies throughout the domain. A symbol that represents an arbitrary number in the *domain* of a function f is called an **independent variable**. A symbol that represents a number in the *range* of f is called a **dependent variable**. In Example A, for instance, r is the independent variable and A is the dependent variable.

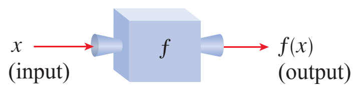

FIGURE 2
Machine diagram for a function f

It's helpful to think of a function as a **machine** (see Figure 2). If x is in the domain of the function f, then when x enters the machine, it's accepted as an **input** and the machine produces an **output** $f(x)$ according to the rule of the function. So we can think of the domain as the set of all possible inputs and the range as the set of all possible outputs. The preprogrammed functions in a calculator are good examples of a function as a machine. For example, if you input a number and press the squaring key, the calculator displays the output, the square of the input.

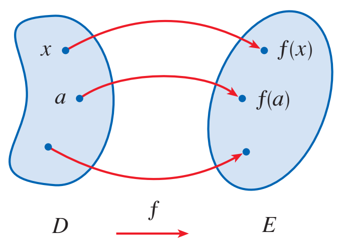

FIGURE 3
Arrow diagram for f

Another way to picture a function is by an **arrow diagram** as in Figure 3. Each arrow connects an element of D to an element of E. The arrow indicates that $f(x)$ is associated with x, $f(a)$ is associated with a, and so on.

Perhaps the most useful method for visualizing a function is its graph. If f is a function with domain D, then its **graph** is the set of ordered pairs

$$\{(x, f(x)) \mid x \in D\}$$

(Notice that these are input-output pairs.) In other words, the graph of f consists of all points (x, y) in the coordinate plane such that $y = f(x)$ and x is in the domain of f.

The graph of a function f gives us a useful picture of the behavior or "life history" of a function. Since the y-coordinate of any point (x, y) on the graph is $y = f(x)$, we can read the value of $f(x)$ from the graph as being the height of the graph above the point x. (See Figure 4.) The graph of f also allows us to picture the domain of f on the x-axis and its range on the y-axis as in Figure 5.

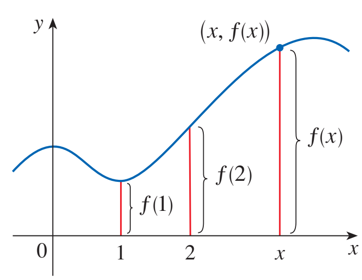

FIGURE 4

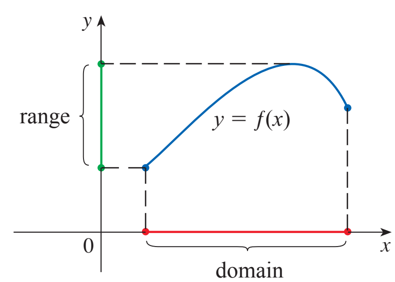

FIGURE 5

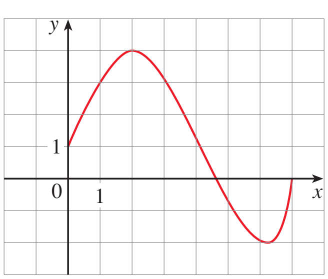

FIGURE 6

EXAMPLE 1 The graph of a function f is shown in Figure 6.
(a) Find the values of $f(1)$ and $f(5)$.
(b) What are the domain and range of f?

SOLUTION
(a) We see from Figure 6 that the point $(1, 3)$ lies on the graph of f, so the value of f at 1 is $f(1) = 3$. (In other words, the point on the graph that lies above $x = 1$ is 3 units above the x-axis.)

When $x = 5$, the graph lies about 0.7 units below the x-axis, so we estimate that $f(5) \approx -0.7$.

(b) We see that $f(x)$ is defined when $0 \leqslant x \leqslant 7$, so the domain of f is the closed interval $[0, 7]$. Notice that f takes on all values from -2 to 4, so the range of f is

$$\{y \mid -2 \leqslant y \leqslant 4\} = [-2, 4]$$ ■

The notation for intervals is given in Appendix A.